(1) First two small observations on the Killing form B.

B is invariant under $A \in \mathrm{Aut}(\mathfrak{g})$.

$A \in \mathrm{Aut}(\mathfrak{g})$ iff $A([x,y]) = [Ax, Ay]$

$\iff A \cdot ad_x = ad_{Ax} \cdot A \iff A \cdot ad_x A^{-1} = ad_{Ax}$.

Hence $B(Ax, Ay) = tr(ad_{Ax} \cdot ad_{Ay})$

$= tr(A\, ad_x A^{-1} \cdot A \cdot ad_y \cdot A^{-1})$

$= tr(A\, ad_x \cdot ad_y A^{-1}) = tr(ad_x \cdot ad_y)$

$= B(x,y)$.

This implies B is invariant under $a \in \mathrm{Der}(\mathfrak{g})$

$\left(a \in \mathrm{Der}(\mathfrak{g}) \iff a([x,y]) = [ax, y] + [x, ay] \right)$

& since $\mathrm{Der}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$.

<u>Corollary</u>: $B(x,y)$. the Killing form is invariant under ad_z

Namely $B(ad_z x, y) + B(x, ad_z y) = 0$

We used the Killing form B is invariant under $ad_{\mathfrak{g}}$ in the last lecture on the structure of semi-simple Lie algebras.

(2) <u>Theorem</u>: (i) $\mathfrak{g}$ is a compact Lie algebra, namely $\mathfrak{g} = \mathrm{Lie}(G)$ for a compact Lie group

then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$ $\quad \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$

& $\mathfrak{g}'$ is semi-simple.

(ii) If $\mathfrak{g}$ is a simple Lie algebra. then $\mathfrak{g}$ is compact $\iff B < 0$.

Pf. If G is compact. $\Longrightarrow$ $\exists\, du$ which is bi-invariant

$\Rightarrow \forall \langle\ \rangle_0$ on $\mathfrak{g}$, define

$$\langle x, y\rangle := \int_G \langle Ad_g(x), Ad_g(y)\rangle_0 \, du$$

$\Rightarrow \langle\ \rangle$ is Ad_g invariant.

$\Rightarrow \langle\ \rangle$ is also $ad_{\mathfrak{g}}$ invariant in the sense

$$\langle ad_x y, z\rangle + \langle y, ad_x z\rangle = 0$$

Using the invariant inner product (which is always non-degenerate).

$\Rightarrow ad_x \in so(\mathfrak{g})$ - namely it is skew symmetric.

$\Longrightarrow$ If ad_x has matrix form (A_{ij})

$$B(x, x) = tr(ad_x \cdot ad_x) = \sum_i \sum_{j=1}^{n} A_{ij} A_{ji}$$

$$= -\sum A_{ij}^2 < 0 \quad \text{unless } A \equiv 0 \Leftrightarrow ad_x = 0$$

Hence we have showed $B(x, x) < 0$ unless $ad_x = 0 \Leftrightarrow x \in \mathfrak{z}(\mathfrak{g})$.

Now consider $\forall\, y\ z \in \mathfrak{g}$.

$$\langle [x, y], z\rangle - \langle x, [y, z]\rangle = 0$$

$$\Leftrightarrow \langle [x, y], z\rangle = 0 \Leftrightarrow \langle x, [y\, z]\rangle = 0$$

$$\Rightarrow x \in (\mathfrak{g}')^{\perp} \Leftrightarrow x \in \mathfrak{z}(\mathfrak{g})$$

$$\Rightarrow \mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$$

Since if $x \in \mathfrak{g}' \Rightarrow$, $ad_x \neq 0 \Rightarrow B(x, x) < 0 \Rightarrow \mathfrak{g}'$ is semi-simple.

For (ii) One direction is easy.

If $\mathfrak{g}$ is simple & compact $\Rightarrow$ $\mathfrak{z}(\mathfrak{g}) \neq \mathfrak{g}$ (Since $\mathfrak{g}$ is NOT Abelian)

But $\mathfrak{z}(\mathfrak{g})$ is an idea. & $\Rightarrow$ $\mathfrak{z}(\mathfrak{g}) = 0$ $\Rightarrow$ $\mathfrak{g} = \mathfrak{g}'$ &

$B|_{\mathfrak{g}'} < 0$, by (i).

For the other direction. $\mathfrak{g}$ is Simple $\Rightarrow$ $\mathrm{Aut}(\mathfrak{g}) / \mathrm{Int}(\mathfrak{g})$ is discrete. But on $\mathrm{Aut}(\mathfrak{g})$, $B < 0$ $\Rightarrow$ $-B$ is an $\mathrm{Aut}(\mathfrak{g})$ invariant, namely $B(Ax, Ay) = B(x, y)$.

$\Rightarrow$ $A \in \mathrm{Aut}(\mathfrak{g})$ is $\in O(\mathfrak{g})$. (with respect to $-B$).

$\Rightarrow$ $\mathrm{Aut}(\mathfrak{g}) \subset O(\mathfrak{g})$.

$\mathrm{Aut}(\mathfrak{g})$ clearly is closed since $A([x, y]) = [Ax, Ay]$ is a close condition $\Rightarrow$ $\mathrm{Aut}(\mathfrak{g})$ is compact $\Rightarrow$ $\mathrm{Aut}(\mathfrak{g}) / \mathrm{Int}(\mathfrak{g})$ must be finite.

$\Rightarrow$ $\mathrm{Int}(\mathfrak{g})$ ~~is~~ compact. $\Rightarrow$ $\mathrm{int}(\mathfrak{g}) \cong \mathfrak{g}$ (due to $\ker \mathrm{ad} = \mathfrak{z}(\mathfrak{g}) = 0$)

$\Rightarrow$ $\mathrm{Int}(\mathfrak{g})$ is a compact Lie group with $\mathfrak{g}$ as its Lie algebra.

<u>Corollary</u>: $\mathfrak{g}$ is compact $\Rightarrow$ $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$

$\mathfrak{g}_i$ compact simple.

③ Weyl's theorem:

(i). A compact Lie group with finite center must be Semi-simple.

(ii). $\forall$ G, Lie group. $\mathfrak{g}$ its Lie algebra.

If $\mathfrak{g}$ is semi-simple & $B|_{\mathfrak{g}} < 0$ $\Longrightarrow$ G must be compact.
[Hence $\tilde{G}$ is compact as well].

Defn: G is called Semi-simple (nilpotent, Soluable) iff $\mathfrak{g}$ is Semi-simple (nilpotent, soluable).

Pf: (i) $Z(G)$ Center of G
$\mathfrak{z}(\mathfrak{g}) = \mathrm{Lie}(Z(G)_0)$ | proved before

$\Longrightarrow$ $Z(G)$ discrete $\Rightarrow$ $\mathfrak{z}(\mathfrak{g}) = 0$ $\Longrightarrow$ $\mathfrak{g}$ is Semi-Simple.

(ii) $\exists$ Algebraic proof.
We present a Riemannian geometric proof.

Bonnet-Meyer Theorem: (M, g) a Riemannian mfld. Complete. & $\mathrm{Ric}_g \geq \lambda g$ (λ constant >0). Ric_g stands for the Ricci curvature of g

$$\Longrightarrow \quad \mathrm{Diam}(M) \leq \frac{\pi}{\sqrt{\lambda}}$$

In particular, M is compact.

$$\mathrm{Ric}(x, y) = \sum_{i=1}^{n} R(e_i, x, y, e_i) \qquad \{e_i\} \text{ orthonormal basis.}$$

R: Curvature Tensor.

The proof of (ii) is to use BM, by calculating the Ricci Curvature of a Bi-invariant metric. [See calculation pdf for details]